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CRACK PATHS IN PLANE SITUATIONS—I. GENERAL FORM OF THE EXPANSION OF THE STRESS INTENSITY FACTORS

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Abstract—The aim of this series of papers is to provide formulas for the geometrical parameters (branching angle, curvature) of a crack propagating in the most general plane situation. These formulas can be used for numerical predictions of crack paths.

The first paper addresses the problem of establishing the *general form* of the first three terms of the expansion of the stress intensity factors in powers of the crack extension length, i.e. of specifying the geometrical and mechanical parameters they depend upon. The treatment is based on two main elements: dimensional analysis (scale changes) and regularity properties (continuity, differentiability) of the stresses with respect to the crack extension length. It is shown that most terms have *universal* expressions in the sense that they depend only on the parameters characterizing the local geometry of the crack and its extension and the stress field near the initial crack tip, whatever the geometry of the body under consideration and the prescribed forces or displacements.

1. INTRODUCTION

Predicting crack paths is a popular problem in LEFM. Most works devoted to this question are restricted to the simpler case of two-dimensional situations (plane strain conditions).

It is clear that the prediction must necessarily be numerical at some stage. Indeed it requires the knowledge of the stress intensity factors (SIFs) at the tip of the propagating crack, and there is no hope that an analytical, explicit formula will ever provide the SIFs in the most general plane situation (arbitrary geometry of the body and the crack, arbitrary loading).

The simplest approach consists in modelling the crack as a succession of straight segments. At each step, once the SIFs at the present crack tip have been evaluated numerically, the branching angle must be deduced from some propagation criterion, for instance the "maximum hoop stress criterion" (Erdogan and Sih, 1963) or the "principle of local symmetry" (PLS) (Goldstein and Salganik, 1974). This method has been used notably by Murakami (1980) and Swenson and Ingraffea (1987). Even in this simple approach, a number of theoretical problems remain unsolved. For instance, no decisive argument has been put forward up to now with regard to the choice which should be made among existing propagation criteria. Moreover, the use of some of these, notably the PLS, requires knowledge of the SIFs just after the kink, and the formulas expressing these SIFs in terms of those just before the kink and the branching angle have been established only in a very particular case: infinite body, uniform forces at infinity, straight initial crack (Bilby *et al.*, 1977; Wu, 1978a, b; Amestoy *et al.*, 1979). However these theoretical difficulties have almost no practical consequences; it is known for instance that all existing criteria lead to very similar numerical predictions. This approach can therefore be rated as operational.

In a more sophisticated approach, the crack is modelled as a succession of curved arcs, with or without kink angles between them. This has been done by Sumi (1986a, b), using the PLS as a criterion. It is then necessary to specify, at each step of the numerical procedure, not only the branching angle (if there is one) but also the curvature of the crack extension. This is achieved by using an analytical expansion of the SIFs in powers of the length s of the crack extension, where the influence of the curvature of the extension appears in an explicit way, and deducing the value of this parameter from the condition that the SIF

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corresponding to mode 2 must vanish along the propagation path, as required by the PLS. This raises the problem of obtaining such an expansion, or at least its first terms.

This theoretical problem is obviously difficult and has received only partial solutions up to now. These solutions can be divided into two classes. The first is composed of solutions which are rigorously exact but valid only for particular geometries and loadings. Chatterjee (1975), using Muskhelishvili's method, considered for instance the problem of a crack consisting of two straight branches in an infinite medium loaded by uniform forces at infinity. The length s of the secondary branch was arbitrary so that the results obtained (which were expressed in a purely numerical form) contained implicitly all the terms of the expansion of the SIFs in powers of s; but this expansion was obviously of limited applicability. The second term (proportional to $\sqrt{s}$) of the expansion of the SIFs was studied in the same particular case by Bilby and Cardew (1975), using the previous work of Khrapkov (1971), and later by Amestoy *et al.* (1986), by a different method and in a more complete way.

The second class is composed of approximate expansions for nearly straight cracks. Cotterell and Rice (1980) used a perturbation method devised by Banichuk (1970) to study a nearly straight crack in an infinite body. The work of Karihaloo *et al.* (1981) was restricted to the case of a straight initial crack, but these authors carried out their analysis to a higher order with respect to the small parameters characterizing the deviations of the crack extension from straightness. Sumi *et al.* (1983) also considered a nearly straight extension of a straight initial (edge) crack, but in a body of arbitrary geometry. Their work is of considerable interest since among the few available expansions, theirs is the only one which applies to bodies of finite dimensions. It was this expansion that Sumi (1986a, b) used for numerical predictions of crack paths. It is still however of limited applicability, and its use in cases where its conditions of validity are not fulfilled certainly leads to errors, though these have not been quantified. This remark applies in particular to Sumi's work (1986a, b).

The aim of the present work is to obtain the beginning of the expansion of the SIFs in powers of the length s of the (kinked and curved) crack extension, in the most general case: arbitrary geometry of the body, the crack and its extension, with arbitrary loading. We will restrict our attention to the first three terms of this expansion (proportional respectively to $s^0 = 1$, $s^{1/2}$ and $s^1 = s$) because, as will appear, this is sufficient to obtain the expression of the curvature at all points of the propagation path. However, extending the analysis to higher orders in s would not raise any difficulty of principle; one would then obtain the expression of the derivative of the curvature with respect to s, of its second derivative, and so on.

It is improbable that analytical methods can solve very general problems. Therefore our objective requires the use of new methods which should not be completely analytical in nature. The approach adopted here is as follows.

We will start by establishing, in this paper, the *general form* of the successive terms of the expansion of the SIFs in an arbitrary situation; this means specifying the various geometrical and mechanical parameters they depend upon. The arguments used will be of very general nature, based essentially on dimensional analysis (scale changes) and on regularity properties (continuity, differentiability) of the stresses at a fixed point with respect to s.

In further papers we will identify the various functions appearing in the expansion by considering some special cases where the solution can be obtained by analytical means. The problem of the propagation criterion will then be studied. It will be shown notably that purely logical considerations of internal coherence within the linear elastic model lead to the PLS as the only possible criterion. We will conclude by combining this criterion with the expansion of the SIFs derived previously to derive formulas for the geometric parameters of the crack extension, which can be used in numerical applications.

This paper is organized as follows. After having stated general hypotheses and notations in Section 2, we establish in Section 3 the continuity of the displacement and stresses at a fixed point of the body when a kink occurs on the crack. This property is used in Section 4 to show that the SIFs just after the kink depend only on those just before the kink and the branching angle, whatever the geometry of the body, the crack and its extension and the prescribed forces or displacements; this means that the formulas established by

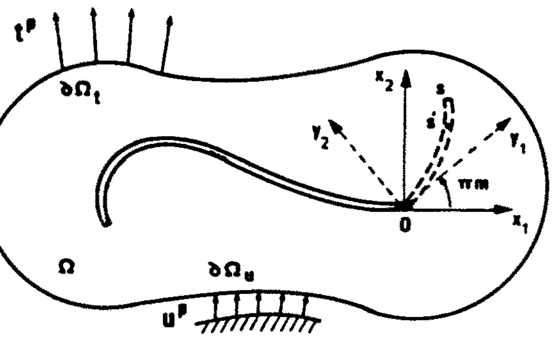

Fig. 1. Definition of the general problem studied.

Bilby *et al.* (1977), Wu (1978a, b) and Amestoy *et al.* (1979) in a particular case are in fact of *universal* value. We refine in Section 5 the results of Section 3 by showing that the displacement and stresses at a fixed point are differentiable with respect to the length s of the (kinked and curved) crack extension at $s = 0$, and that the corresponding derivatives are independent of the curvature of the extension. These properties are used in Sections 6 and 7 to study the second and third terms of the expansion of the SIFs (proportional to $s^{1/2}$ and $s^1 = s$). These terms are expressed as sums of quantities, most of which are again of *universal* character, in the sense that they depend only on *local* parameters describing the geometry of the crack and its extension (branching angle, curvature parameters) and the *initial* stress field (SIFs, non-singular stress...), without any explicit reference to the far geometry of the body nor to the loading imposed on its boundary. However the third term of the expansion is shown to involve one *non-universal* quantity, which depends on the geometry of the entire body under consideration and must therefore be evaluated in each particular case. Finally we outline in conclusion a numerical method for crack path predictions based on these results.

2. STATEMENT OF THE PROBLEM

We consider (Fig. 1) the general problem of an elastic body Ω under plane strain conditions, containing a curvilinear crack. The boundary of this body, including the lips of the crack, is subjected to *constant* (with respect to time)† prescribed line tractions $\mathbf{t}^P$ on a portion $\partial\Omega_t$ and to *constant* prescribed displacements $\mathbf{u}^P$ on the complementary part $\partial\Omega_u$. The lips of the crack are supposed to be traction-free in the vicinity of the crack tip and to remain traction-free upon subsequent propagation; we exclude thus the case of a crack loaded by an internal pressure due to a fluid. There are no body forces.

At the instant considered, the crack extends up to a point O, where its curvature is C. The subsequent crack extension makes an initial angle πm $(-1 < m < +1)$ with the tangent Ox_1 to the crack at the point O. (The case of regular propagation with a continuously varying tangent will be treated as a particular case where m takes the value 0). The length of the extension will be denoted s and the distance along the extension from the point O to an arbitrary point, s'. Oy_1y_2 being an orthonormal coordinate system with first axis directed along the tangent to the extension at the point O, the equation of the extension will be supposed to be of the form

$$y_2 = a^* y_1^{3/2} + \frac{C^*}{2} y_1^2 + o(y_1^2) \tag{1}$$

where a^* and C^* are parameters.‡ It is proved in Appendix A that crack extensions of that

† In fact, except in the case of subcritical propagation, (quasistatic) propagation of the crack is possible only if the load applied varies. This variation will be introduced in the subsequent papers; it will be shown notably that for *proportional* loadings, it has no influence on the propagation path.

‡ The notations O and o are used throughout this paper. It is recalled that a function is $O(x^a)$ if it is bounded by some constant times x^a for $x \to 0$, and $o(x^a)$ if it is of the form $x^a f(x)$ with $\lim_{x \to 0} f(x) = 0$.

shape must necessarily be considered, if these extensions are to be obtainable by actual propagation of the crack and not simply by arbitrary machining of the body, and if propagation is to obey the PLS.

The general object of our study is the expansion of the SIFs $k_1(s)$, $k_2(s)$ at the tip of the extended crack in powers of s.

3. CONTINUITY OF THE DISPLACEMENT AND STRESSES AT A FIXED POINT WITH RESPECT TO THE CRACK EXTENSION LENGTH

The aim of this section is to establish that *at a fixed point M of the body, the stresses are continuous with respect to s at $s = 0$*, i.e. they do not undergo a sudden jump when the kink occurs. In fact our proof will apply to the displacement as well as to the stresses.

We consider the body in two situations. In the first one, the crack ends up at the point O, and the displacement at M is $\mathbf{u}(M)$. The crack extension can be supposed to be opened over a length s provided suitable tractions $\mathbf{t}^{\pm}(s')$ are exerted on its upper $(+)$ and lower $(-)$ lips; these tractions are $O(s'^{-1/2})$. In the second situation, the crack is extended further over a length s, i.e. the tractions just mentioned are released. The displacement at M is then $\mathbf{u}(M, s)$.

Taking the difference between these two situations, we obtain what will be called problem A. In this problem a zero traction is imposed on $\partial\Omega_t$, a zero displacement on $\partial\Omega_u$, and tractions $-\mathbf{t}^{\pm}(s')$ are exerted on the lips of the crack extension. The displacement at M is $\mathbf{u}(M, s) - \mathbf{u}(M)$.

$(\mathbf{e}_1, \mathbf{e}_2)$ being an orthonormal basis, we define a problem B as follows: the crack extends over a length s from the point O; a zero traction is imposed on $\partial\Omega_t$, a zero displacement on $\partial\Omega_u$, and a unit point force in the direction $\mathbf{e}_i$ is exerted on the point M. The resulting displacements on the lips of the crack extension are denoted $\mathbf{v}^{(i)\pm}(M, s, s')$.

Application of Betti's theorem to problems A and B yields

$$u_i(M, s) - u_i(M) = -\int_0^s [\mathbf{t}^+(s') \cdot \mathbf{v}^{(i)+}(M, s, s') + \mathbf{t}^-(s') \cdot \mathbf{v}^{(i)-}(M, s, s')] \, \mathrm{d}s'.$$

Differentiating this equation with respect to the coordinates x_j of M, we get also

$$\frac{\partial u_i}{\partial x_j}(M, s) - \frac{\partial u_i}{\partial x_j}(M) = -\int_0^s \left[\mathbf{t}^+(s') \cdot \frac{\partial \mathbf{v}^{(i)+}}{\partial x_j}(M, s, s') + \mathbf{t}^-(s') \cdot \frac{\partial \mathbf{v}^{(i)-}}{\partial x_j}(M, s, s') \right] \mathrm{d}s'.$$

The quantities $(\partial \mathbf{v}^{(i)\pm}/\partial x_j)(M, s, s')$ in this equation can be interpreted as the displacements on the lips of the crack induced by a unit "dipole" at M, i.e. two infinite opposite forces parallel to $\mathbf{e}_i$ applied on points separated by an infinitesimal vector collinear to $\mathbf{e}_j$, the distance between the points times the intensity of the forces being equal to unity.

Let A and B be upper bounds for $|\mathbf{v}^{(i)\pm}(M, s, s')|$ and $|\partial \mathbf{v}^{(i)\pm}/\partial x_j(M, s, s')|$.† Then

$$|u_i(M, s) - u_i(M)| \leqslant A \int_0^s (|\mathbf{t}^+(s')| + |\mathbf{t}^-(s')|) \, \mathrm{d}s';$$

$$\left| \frac{\partial u_i}{\partial x_j}(M, s) - \frac{\partial u_i}{\partial x_j}(M) \right| \leqslant B \int_0^s (|\mathbf{t}^+(s')| + |\mathbf{t}^-(s')|) \, \mathrm{d}s'.$$

† $\mathbf{v}^{(i)\pm}(M, s, s')$ and $\dfrac{\partial \mathbf{v}^{(i)\pm}}{\partial x_j}(M, s, s')$ are obviously bounded functions of s' for every s, i.e.

$$|\mathbf{v}^{(i)\pm}(M, s, s')| \leqslant C_1(s), \quad \left| \frac{\partial \mathbf{v}^{(i)\pm}}{\partial x_j}(M, s, s') \right| \leqslant C_2(s) \text{ for all } s', 0 < s' < s;$$

it is implicitly assumed here that for sufficiently small values of s, they are in fact bounded functions of *both s' and s*, i.e. $C_1(s)$ and $C_2(s)$ are bounded functions of s. The opposite would mean that when the length s of the crack extension is shrunk to zero, some part of this extension goes to infinity under the effect of the point force or the "dipole" at M!

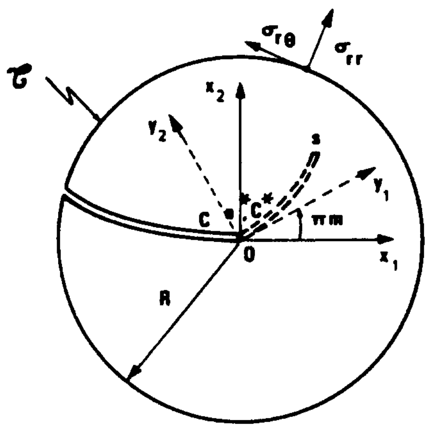

Fig. 2. Edge crack in a circular disk.

Since $t^{\pm}(s')$ is $O(s'^{-1/2})$, these inequalities imply that $|u_i(M,s)-u_i(M)|$ and $|(\partial u_i/\partial x_j)\times(M,s)-(\partial u_i/\partial x_j)(M)|$ are $O(\sqrt{s})$. Therefore the displacement and its gradient (and hence the stresses) are continuous with respect to s at $s = 0$.

It must be stressed that this continuity property is satisfied only because the displacement and the stresses are considered at a point the position of which is fixed *independently of that of the crack tip*. Quantities such as the SIFs, which characterize the stress field near the moving singularity, are known to be discontinuous at $s = 0$ when the crack extension makes a non-zero angle with the initial crack.

4. STRESS INTENSITY FACTORS JUST AFTER THE KINK

We will now study the limit $\mathbf{k}^* = (k_1^*, k_2^*)$ of $\mathbf{k}(s) = [k_1(s), k_2(s)]$ when s tends towards zero. More specifically, it will be shown that k_1^* and k_2^* depend only on the SIFs k_1, k_2 just before the kink and the branching angle πm. This means in fact extending the validity of the formulas for the k_p^*s established by Bilby *et al.* (1977), Wu (1978a, b) and Amestoy *et al.* (1979) in a particular case (straight initial crack, straight extension, infinite body, uniform forces at infinity) to fully general situations.

We suppose first that the body is a circular disk of centre O, of radius R, subjected to prescribed *tractions* and containing a *traction-free edge* crack (Fig. 2). r and θ denoting polar coordinates with respect to the Ox_1 axis, let $\mathbf{t}(\theta) = [\sigma_{rr}(\theta), \sigma_{r\theta}(\theta)]$ be the traction prescribed and $\mathscr{F} = \{\mathbf{t}(\theta)\}$ the force field defined by this traction. The SIFs at the tip of the extended crack are a continuous functional of this force field and all the geometric parameters of the problem, linear with respect to $\mathscr{F}$. This can be written symbolically

$$\mathbf{k}(s) = \mathscr{L}(m, R, C, a^*, C^*, s, \mathscr{F}), \tag{2}$$

omitting for simplicity indications of dependence upon the remaining geometric parameters, namely the derivatives of the curvature of the main crack at the point O and parameters characterizing the crack extension shape to higher degrees of accuracy than a^* and C^*. (That this omission is valid will be shown below.)

Let us consider a new structure identical to the first one, except that all dimensions are multiplied by a factor λ; the geometric parameters m, R, C, a^*, C^*, s become m, λR, C/λ, $a^*/\sqrt{\lambda}$, C^*/λ, λs. Let the new structure be subjected to the same force field $\mathscr{F}$ as the old one, i.e. two points having the same polar angle are subjected to identical forces per unit length. The stresses are then the same at homothetical points so that the SIFs, which are limits of certain stress components times the square root of the distance from the crack tip, are $\sqrt{\lambda}$ greater in the new structure than in the old one. Thus the functional $\mathscr{L}$ verifies the following "homogeneity property" with respect to the geometric parameters:

$$\mathscr{L}(m, \lambda R, C/\lambda, a^*/\sqrt{\lambda}, C^*/\lambda, \lambda s, \mathscr{F}) = \sqrt{\lambda}\mathscr{L}(m, R, C, a^*, C^*, s, \mathscr{F}). \tag{3}$$

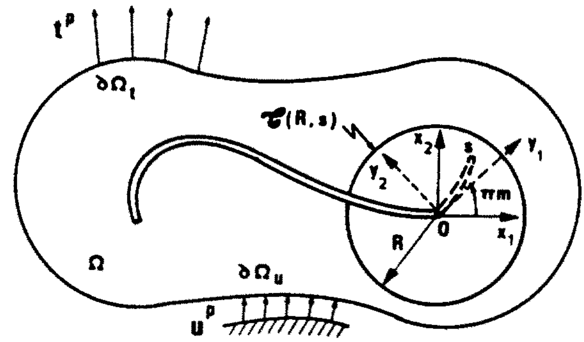

Fig. 3. Circular disk centered at the crack tip in an arbitrary body.

Let $\mathscr{L}^*$ be the limit of the functional $\mathscr{L}$ when s tends towards zero (this is the functional that gives $\mathbf{k}^*$). Taking the limit $s \to 0$ in eqn (3), it is easily shown that $\mathscr{L}^*$ satisfies the same homogeneity property as $\mathscr{L}$:

$$\mathscr{L}^*(m, \lambda R, C/\lambda, a^*/\sqrt{\lambda}, C^*/\lambda, \mathscr{I}) = \sqrt{\lambda}\mathscr{L}^*(m, R, C, a^*, C^*, \mathscr{I}). \tag{4}$$

We now come back to a body of arbitrary shape (Fig. 3). We consider, within the body, circular disks of centre O and sufficiently small radius R for the crack to intersect their boundary and to be traction-free within them (this is possible since the crack is supposed to be traction-free in the vicinity of its tip: see Section 2). Let $\mathscr{I}(R, s)$ be the force field on the boundary of the disk of radius R which results from the application of the prescribed tractions $\mathbf{t}^p$ and displacements $\mathbf{u}^p$ on $\partial\Omega_t$ and $\partial\Omega_u$, when the crack extension length is s. The SIFs are unchanged if one eliminates the exterior of the disk of radius R while exerting the force field $\mathscr{I}(R, s)$ on its boundary. Therefore they can be expressed, using definition (2) of $\mathscr{L}$, as

$$\mathbf{k}(s) = \mathscr{L}[m, R, C, a^*, C^*, s, \mathscr{I}(R, s)]. \tag{5}$$

We now let s tend towards zero in this equation, R being fixed. Then $\mathscr{L}$ tends towards $\mathscr{L}^*$ and $\mathscr{I}(R, s)$ tends towards the stress field $\mathscr{I}(R)$ exerted on the boundary of the disk of radius R *before* the kink, because of the property of continuity of the stresses at a fixed point established in Section 3. Therefore eqn (5) becomes:

$$\mathbf{k}^* = \lim_{s \to 0} \mathbf{k}(s) = \mathscr{L}^*[m, R, C, a^*, C^*, \mathscr{I}(R)]. \tag{6}$$

Note the remarkable property that the SIFs *just after* the kink depend only on the stress field *before* the kink. It remains to show that they depend on it only through the initial SIFs.

Using the homogeneity property of $\mathscr{L}^*$ (eqn (4)) with $\lambda = 1/R$ and the linearity with respect to the force field, we transform eqn (6) into

$$\mathbf{k}^* = \mathscr{L}^*[m, 1, RC, \sqrt{R}a^*, RC^*, \sqrt{R}\mathscr{I}(R)]. \tag{7}$$

In intuitive terms, this transformation corresponds to looking at the disk of radius R through a magnifying glass. Now let $R \to 0$. The traction $\mathbf{t} = (\sigma_{rr}, \sigma_{r\theta})$ on the boundary of the disk of radius R admits an expansion of the form

$$\mathbf{t} = k_p \frac{\mathbf{f}_p(\theta)}{\sqrt{R}} + T\mathbf{g}(\theta) + [b_p \mathbf{h}_p(\theta) + Ck_p \tilde{\mathbf{h}}_p(\theta)]\sqrt{R} + O(R) \tag{8}$$

where k_1, k_2, T, b_1, b_2 are coefficients (k_1, k_2 and T are the *initial* SIFs and non-singular stress). This is the classical Irwin–Williams stress expansion for a straight crack except for the corrective term $Ck_p\tilde{\mathbf{h}}_p(\theta)\sqrt{R}$ due to curvature; the existence of this corrective term and the detailed expression of the functions $\tilde{\mathbf{h}}_p(\theta)$ involved are established in the work of Ting (1985). Therefore

$$\sqrt{R}\mathscr{I}(R) = \{\sqrt{R}\mathbf{t}\} = k_p\{\mathbf{f}_p(\theta)\} + O(\sqrt{R}).$$

Furthermore the functional $\mathscr{L}^*(m, 1, RC, \sqrt{R}a^*, RC^*, \cdot)$ tends towards $\mathscr{L}^*(m, 1, 0, 0, 0, \cdot)$ when R tends towards zero. Therefore eqn (7) becomes in this limit:

$$\mathbf{k}^* = \mathscr{L}^*[m, 1, 0, 0, 0, k_p\{\mathbf{f}_p(\theta)\}] = \mathbf{F}(m)\cdot\mathbf{k} \tag{9}$$

where $\mathbf{F}(m)$ is a linear operator depending only on m and $\mathbf{k}$ the vector (k_1, k_2).

It is observed that all curvature parameters vanish in this final expression; in intuitive terms, this is because the crack and its extension appear as straight in the limit of infinitesimal disks. It is easy to see that the geometric parameters omitted for simplicity in (2) would vanish in the final result as well, were they included in the notation; for instance the first derivative of the curvature of the main crack at the point O would appear multiplied by R^2 in (7) and consequently vanish in (9).

Equation (9) establishes the result announced. The above proof is the first one which applies to fully general situations; other authors have given proofs under more or less restrictive hypotheses: Cotterell and Rice (1980) for a nearly straight initial crack with a small branching angle in an infinite body, and Sumi *et al.* (1983) for a straight initial edge crack with a small branching angle in a body of arbitrary geometry.

An expression such as (9) will be termed *universal* in the sense that it depends on the geometry and the loading only through *local* parameters characterizing the crack shape near the initial crack tip and the *initial* stress field (here the branching angle and the initial SIFs). In contrast, a *non-universal* expression will depend on the geometry of the entire body considered and/or on the whole traction and displacement fields imposed on its boundary, and will therefore require a specific evaluation in each particular case.

5. DIFFERENTIABILITY OF THE STRESSES AT A FIXED POINT WITH RESPECT TO THE CRACK EXTENSION LENGTH

The object of this section is to show that the displacement $\mathbf{u}(M, s)$ and stresses $\boldsymbol{\sigma}(M, s)$ at a fixed point M are differentiable with respect to s at $s = 0$, and that the corresponding derivatives $\partial\mathbf{u}/\partial s$ $(M, s = 0)$ and $\partial\boldsymbol{\sigma}/\partial s$ $(M, s = 0)$ are independent of the curvature parameters a^*, C^* of the crack extension.

We will use the following classical mathematical result:

Proposition. Let f be a real function of a real variable x, defined for $x \geqslant 0$, continuous at $x = 0$, differentiable for $x > 0$ and such that $f'(x)$ tends toward a limit l when x tends towards zero. Then f admits a (right-hand) derivative equal to l at $x = 0$.

For every $s > 0$, the crack propagates regularly, i.e. with a continuously varying tangent. Therefore Rice's formulation of the theory of Bueckner's weight functions (see for example Rice, 1985) can be used to evaluate the derivative $\partial u_i/\partial s$ (M, s), yielding

$$\frac{\partial u_i}{\partial s}(M, s) = \frac{2(1-\nu^2)}{E}k_p(s)\tilde{k}_{ip}(M, s). \tag{10}$$

In this equation E and ν are Young's modulus and Poisson's ratio and $\tilde{k}_{ip}(M, s)$ denotes the pth SIF at the tip of the crack extension of length s which results from the application of a unit point force in the direction $\mathbf{e}_i$ at the point M, the portions $\partial\Omega_t$ and $\partial\Omega_u$ of the boundary of the body being simultaneously subjected respectively to a zero traction and a zero displacement. Differentiating eqn (10) with respect to the coordinates of M, we also obtain

$$\frac{\partial}{\partial s}\left(\frac{\partial u_i}{\partial x_j}\right)(M,s) = \frac{2(1-\nu^2)}{E} k_p(s) \frac{\partial \tilde{k}_{ip}}{\partial x_j}(M,s). \tag{11}$$

The quantity $\partial\tilde{k}_{ip}/\partial x_j\ (M,s)$ in (11) can be interpreted in the same way as $\tilde{k}_{ip}(M,s)$, replacing the unit point force exerted on M by a unit "dipole" like in Section 3. The right-hand sides of eqns (10) and (11) tend towards the limits $(2(1-\nu^2)/E)k_p^*\tilde{k}_{ip}^*(M)$ and $(2(1-\nu^2)/E)k_p^*\partial\tilde{k}_{ip}^*/\partial x_j(M)$ when s tends towards zero. Furthermore $u_i(M,s)$ and $\partial u_i/\partial x_j \times (M,s)$ were proved in Section 3 to be continuous with respect to s at $s = 0$. Using the above proposition, we conclude that $u_i(M,s)$ and $\partial u_i/\partial x_j\ (M,s)$ (and hence the stresses) are differentiable with respect to s at $s = 0$, and that

$$\frac{\partial u_i}{\partial s}(M,s=0) = \frac{2(1-\nu^2)}{E} k_p^* \tilde{k}_{ip}^*(M) \tag{12}$$

$$\frac{\partial}{\partial s}\left(\frac{\partial u_i}{\partial x_j}\right)(M,s=0) = \frac{2(1-\nu^2)}{E} k_p^* \frac{\partial \tilde{k}_{ip}^*}{\partial x_j}(M). \tag{13}$$

It is now easy to show that $\partial u_i/\partial s\ (M,s=0)$ and $\partial/\partial s(\partial u_i/\partial x_j)\ (M,s=0)$ (and hence $\partial\sigma/\partial s\ (M,s=0)$) are independent of a^* and C^*. Indeed the SIFs before the kink are independent of the curvature parameters a^*, C^* of the *future* crack extension; since the SIFs just after the kink have been shown in Section 4 to depend only on those before the kink and the branching angle, they are also independent of a^* and C^*. The same argument applies to the $\tilde{k}_{ip}^*(M)$s and $\partial\tilde{k}_{ip}^*/\partial x_j\ (M)$s. Equations (12) and (13) imply then that $\partial u_i/\partial s\ (M,s=0)$ and $\partial/\partial s(\partial u_i/\partial x_j)\ (M,s=0)$ are also independent of a^* and C^*. This concludes the proof.

It is worth noting that $\mathbf{u}(M,s)$ and $\boldsymbol{\sigma}(M,s)$ are *not* twice differentiable with respect to s at $s = 0$. Indeed, anticipating (see Section 6 below) that the expansion of $\mathbf{k}(s)$ contains a term proportional to $\sqrt{s}$, one notes that the expressions (10, 11) of $\partial u_i/\partial s\ (M,s)$ and $\partial/\partial s(\partial u_i/\partial x_j)\ (M,s)$ contain also such terms. This shows clearly—if necessary—that the regularity properties of $\mathbf{u}$ and $\boldsymbol{\sigma}$ studied in Section 3 and here cannot be simply accepted as "intuitively evident" and need to be established in a rigorous way.

6. SECOND TERM OF THE EXPANSION OF THE STRESS INTENSITY FACTORS IN POWERS OF THE CRACK EXTENSION LENGTH

The second term of the expansion $\mathbf{k}(s)$ in powers of s can be studied by the same kind of method as the first one, carrying all expansions up to order $\sqrt{s}$ instead of $s^0 = 1$.

It is proved in Appendix A that the expansion of the functional $\mathscr{L}$ in powers of s does not contain any term proportional to s^β with $0 < \beta < 1/2$ (this is in fact a consequence of the crack extension shape, as described by eqn (1)). Thus it is of the form

$$\mathscr{L}(m,R,C,a^*,C^*,s,\cdot) = \mathscr{L}^*(m,R,C,\cdot) + \mathscr{L}^{(1/2)}(m,R,C,a^*,C^*,\cdot)\sqrt{s} + o(\sqrt{s}) \tag{14}$$

where the arguments a^* and C^* have been omitted in the functional $\mathscr{L}^*$ since $\mathbf{k}^*$ (and hence $\mathscr{L}^*$) have been shown in Section 4 to be independent of these parameters.

Using (14) to expand (3) in powers of s, we get

$$\mathscr{L}^*(m,\lambda R,C/\lambda,\mathscr{I}) + \mathscr{L}^{(1/2)}(m,\lambda R,C/\lambda,a^*/\sqrt{\lambda},C^*/\lambda,\mathscr{I})\sqrt{\lambda s} + o(\sqrt{s})$$
$$= \sqrt{\lambda}\mathscr{L}^*(m,R,C,\mathscr{I}) + \sqrt{\lambda}\mathscr{L}^{(1/2)}(m,R,C,a^*,C^*,\mathscr{I})\sqrt{s} + o(\sqrt{s}).$$

Identification of the terms of order $\sqrt{s}$ in both sides of this equality yields the following homogeneity property for $\mathscr{L}^{(1/2)}$:

$$\mathscr{L}^{(1/2)}(m,\lambda R,C/\lambda,a^*/\sqrt{\lambda},C^*/\lambda,\mathscr{I}) = \mathscr{L}^{(1/2)}(m,R,C,a^*,C^*,\mathscr{I}), \tag{15}$$

which differs from that for $\mathscr{L}^*$ (eqn (4)) by a factor $\sqrt{\lambda}$.

The results of Section 5 imply that the force field $\mathscr{I}(R,s)$ on the boundary of the disk of radius R is differentiable with respect to s at $s = 0$. Therefore its expansion in powers of s is of the form

$$\mathscr{I}(R,s) = \mathscr{I}(R)+\mathscr{I}^{(1)}(R)s+o(s). \tag{16}$$

Using this expression, eqn (14) and the linearity of the functionals with respect to the force field to expand expression (5) for $\mathbf{k}(s)$ in powers of s, we get

$$\mathbf{k}(s) = \mathbf{k}^*+\mathbf{k}^{(1/2)}\sqrt{s}+o(\sqrt{s})$$

where $\mathbf{k}^*$ and $\mathbf{k}^{(1/2)}$ are given respectively by (6) and

$$\mathbf{k}^{(1/2)} = \mathscr{L}^{(1/2)}[m,R,C,a^*,C^*,\mathscr{I}(R)]. \tag{17}$$

Note that $\mathbf{k}^{(1/2)}$, just like $\mathbf{k}^*$, depends only on the stress field $\mathscr{I}(R)$ *before* the kink. This is a direct consequence of the absence of a $\sqrt{s}$ term in the expansion of $\mathscr{I}(R,s)$, i.e. of the differentiability of the stresses with respect to s at $s = 0$.

Using (15) with $\lambda = 1/R$, we transform (17) into

$$\mathbf{k}^{(1/2)} = \mathscr{L}^{(1/2)}[m,1,RC,\sqrt{R}a^*,RC^*,\mathscr{I}(R)].$$

Expanding now the functional $\mathscr{L}^{(1/2)}(m,1,RC,\sqrt{R}a^*,RC^*,\cdot)$ in powers of R and using eqn (8) for $\mathbf{t} = (\sigma_{rr},\sigma_{r\theta})$, we get

$$\mathbf{k}^{(1/2)} = \mathscr{L}^{(1/2)}[m,1,0,0,0,k_p\{\mathbf{f}_p(\theta)\}]\frac{1}{\sqrt{R}}+\mathscr{L}^{(1/2)}[m,1,0,0,0,T\{\mathbf{g}(\theta)\}]$$
$$+a^*\frac{\partial\mathscr{L}^{(1/2)}}{\partial a^*}[m,1,0,0,0,k_p\{\mathbf{f}_p(\theta)\}]+O(\sqrt{R}).$$

This equation holds for every R, which means that the right-hand side is in fact independent of R. Therefore the divergent $R^{-1/2}$ term must be zero. The $O(\sqrt{R})$ term is also zero, because it must be constant while tending towards zero when $R\to 0$. The expression of $\mathbf{k}^{(1/2)}$ becomes thus

$$\mathbf{k}^{(1/2)} = \mathscr{L}^{(1/2)}[m,1,0,0,0,T\{\mathbf{g}(\theta)\}]+a^*\frac{\partial\mathscr{L}^{(1/2)}}{\partial a^*}[m,1,0,0,0,k_p\{\mathbf{f}_p(\theta)\}]$$
$$= T\mathbf{G}(m)+a^*\mathbf{H}(m)\cdot\mathbf{k} \tag{18}$$

where $\mathbf{G}(m)$ and $\mathbf{H}(m)$ are a vector and a linear operator depending only on m. Hence $\mathbf{k}^{(1/2)}$ has a *universal* expression (in the sense defined in Section 4) like $\mathbf{k}^*$.

In the particular case of a straight initial crack with a nearly straight extension (small parameters m, a^*, C^*), Karihaloo *et al.* (1981) and Sumi *et al.* (1983) obtained expressions of $\mathbf{k}^{(1/2)}$ which fit into the general form (18). These expressions appeared in those works as the beginning of an infinite expansion in powers of m, a^*, C^*. Equation (18) indicates that quite remarkably, the only powers of a^* and C^* involved in this expansion are in fact 1 and a^*.

7. THIRD TERM OF THE EXPANSION OF THE STRESS INTENSITY FACTORS IN POWERS OF THE CRACK EXTENSION LENGTH

The functional $\mathscr{L}$ is now expanded up to order $O(s)$:

$$\mathscr{L}(m, R, C, a^*, C^*, s, \cdot) = \mathscr{L}^*(m, R, C, \cdot) + \mathscr{L}^{(1/2)}(m, R, C, a^*, \cdot)\sqrt{s} + \mathscr{L}^{(1)}(m, R, C, a^*, C^*, \cdot)s + o(s) \quad (19)$$

where the argument C^* has been dropped in $\mathscr{L}^{(1/2)}$ since (18) implies that $\mathbf{k}^{(1/2)}$ and $\mathscr{L}^{(1/2)}$ are independent of this parameter. The $O(s)$ dependence of the third term of the expansion can be justified in the same way as the $O(\sqrt{s})$ dependence of the second one (see Appendix A). Expanding eqn (3) up to order $O(s)$ and identifying terms of this order in the resulting identity, we get the following homogeneity property for $\mathscr{L}^{(1)}$:

$$\mathscr{L}^{(1)}(m, \lambda R, C/\lambda, a^*/\sqrt{\lambda}, C^*/\lambda, \mathscr{I}) = \frac{1}{\sqrt{\lambda}}\mathscr{L}^{(1)}(m, R, C, a^*, C^*, \mathscr{I}). \quad (20)$$

We expand now eqn (5) in powers of s, using (16) and (19); we obtain thus

$$\mathbf{k}(s) = \mathbf{k}^* + \mathbf{k}^{(1/2)}\sqrt{s} + \mathbf{k}^{(1)}s + o(s) \quad (21)$$

where

$$\mathbf{k}^{(1)} = \mathscr{L}^*[m, R, C, \mathscr{I}^{(1)}(R)] + \mathscr{L}^{(1)}[m, R, C, a^*, C^*, \mathscr{I}(R)],$$

or equivalently by eqn (20) (with $\lambda = 1/R$):

$$\mathbf{k}^{(1)} = \mathscr{L}^*[m, R, C, \mathscr{I}^{(1)}(R)] + \mathscr{L}^{(1)}\left[m, 1, RC, \sqrt{R}a^*, RC^*, \frac{1}{\sqrt{R}}\mathscr{I}(R)\right]. \quad (22)$$

This equation shows that $\mathbf{k}^{(1)}$, unlike $\mathbf{k}^*$ and $\mathbf{k}^{(1/2)}$, depends on the stress field $\mathscr{I}(R, s)$ *after* the kink, through its derivative $\mathscr{I}^{(1)}(R)$ with respect to s at $s = 0$.

We expand now the functional $\mathscr{L}^{(1)}(m, 1, RC, \sqrt{R}a^*, RC^*, \cdot)$ in powers of R and use eqn (8); eqn (22) becomes

$$\begin{aligned}
\mathbf{k}^{(1)} = {} & \mathscr{L}^*[m, R, C, \mathscr{I}^{(1)}(R)] + \mathscr{L}^{(1)}[m, 1, 0, 0, 0, k_p\{\mathbf{f}_p(0)\}]\frac{1}{R} \\
& + \mathscr{L}^{(1)}[m, 1, 0, 0, 0, T\{\mathbf{g}(0)\}]\frac{1}{\sqrt{R}} + a^*\frac{\partial \mathscr{L}^{(1)}}{\partial a^*}[m, 1, 0, 0, 0, k_p\{\mathbf{f}_p(0)\}]\frac{1}{\sqrt{R}} \\
& + \mathscr{L}^{(1)}[m, 1, 0, 0, 0, b_p\{\mathbf{h}_p(0)\}] \\
& + C\frac{\partial}{\partial C'}[\mathscr{L}^{(1)}(m, 1, C', 0, 0, k_p\{\mathbf{f}_p(0) + C'\tilde{\mathbf{h}}_p(0)\})]_{C'=0} \\
& + a^*\frac{\partial \mathscr{L}^{(1)}}{\partial a^*}[m, 1, 0, 0, 0, T\{\mathbf{g}(0)\}] + \frac{a^{*2}}{2}\frac{\partial^2 \mathscr{L}^{(1)}}{\partial a^{*2}}[m, 1, 0, 0, 0, k_p\{\mathbf{f}_p(0)\}] \\
& + C^*\frac{\partial \mathscr{L}^{(1)}}{\partial C^*}[m, 1, 0, 0, 0, k_p\{\mathbf{f}_p(0)\}] + O(\sqrt{R}). \quad (23)
\end{aligned}$$

The left-hand side of this equation is independent of R. In the right-hand side, all terms from the fifth to the ninth are independent of R, and the last one tends towards zero with R. Hence the sum of the first four terms has a finite limit for $R \to 0$:

$$\mathscr{L}^*[m, R, C, \mathscr{I}^{(1)}(R)] + \mathscr{L}^{(1)}[m, 1, 0, 0, 0, k_p\{\mathbf{f}_p(\theta)\}]\frac{1}{R}$$

$$+\mathscr{L}^{(1)}[m, 1, 0, 0, 0, T\{\mathbf{g}(\theta)\}]\frac{1}{\sqrt{R}} + a^*\frac{\partial\mathscr{L}^{(1)}}{\partial a^*}[m, 1, 0, 0, 0, k_p\{\mathbf{f}_p(\theta)\}]\frac{1}{\sqrt{R}} = O(1).$$

Now (13) implies that $\mathscr{I}^{(1)}(R)$ depends on the loading only through the k_p^*s, i.e. through the k_ps by (9). Hence, in the left-hand side of the above equality, the third term is the only one which depends on T. Since it is divergent, it must be zero. (To see that, vary the value of T in the above equality.) Similarly, the fourth term is the only one which depends on a^*($\mathscr{I}^{(1)}(R)$ is independent of this parameter: see Section 5) and diverges, so it must also be zero. Hence the above equality implies that $\mathscr{L}^*[m, R, C, \mathscr{I}^{(1)}(R)]$ is the sum of a divergent term $\mathscr{L}^{(1)}[m, 1, 0, 0, 0, k_p\{\mathbf{f}_p(\theta)\}]/R$ and a term which has a finite limit for $R \to 0$. This limit will be called the *principal part* of $\mathscr{L}^*[m, R, C, \mathscr{I}^{(1)}(R)]$ (for $R \to 0$) and denoted $\mathbf{Z}$:

$$\mathbf{Z} = \lim_{R\to 0}\left\{\mathscr{L}^*[m, R, C, \mathscr{I}^{(1)}(R)] + \mathscr{L}^{(1)}[m, 1, 0, 0, 0, k_p\{\mathbf{f}_p(\theta)\}]\frac{1}{R}\right\}. \tag{24}$$

Letting $R \to 0$ in (23) and using (24), we get the final expression of $\mathbf{k}^{(1)}$:

$$\mathbf{k}^{(1)} = \mathbf{Z} + \mathbf{I}(m)\cdot\mathbf{b} + C\mathbf{J}(m)\cdot\mathbf{k} + a^*T\mathbf{K}(m) + a^{*2}\mathbf{L}(m)\cdot\mathbf{k} + C^*\mathbf{M}(m)\cdot\mathbf{k} \tag{25}$$

where $\mathbf{b}$ is the vector (b_1, b_2) and $\mathbf{I}(m)$, $\mathbf{J}(m)$, $\mathbf{K}(m)$, $\mathbf{L}(m)$, $\mathbf{M}(m)$ vectors or linear operators which depend only on m.

The last five terms in the right-hand side of (25) are universal in the sense defined in Section 4. On the other hand a detailed analysis of the $\mathbf{Z}$ term (see Appendix B) shows that this quantity is not universal, because it does not depend only on local geometric parameters but also *on the far geometry of the body considered.*

Equation (25) confirms and extends the results of Sumi *et al.* (1983). These authors were first to note the loss of the universality property in the third term of the expansion of $\mathbf{k}(s)$ in powers of s, in the particular case of a straight initial edge crack with small parameters m, a^*, C^*. They obtained $\mathbf{k}^{(1)}$ as a sum of three universal terms proportional to $\mathbf{b}$, a^*T and $C^*\mathbf{k}$, in accordance with eqn (25) (the term proportional to $C\mathbf{k}$ was absent because of the assumed straightness of the initial crack and that proportional to $a^{*2}\mathbf{k}$ because the treatment was limited to the first order in m, a^*, C^*), plus a non-universal one $\mathbf{Z}$. The latter quantity was not interpreted as the principal part of $\mathscr{L}^*[m, R, C, \mathscr{I}^{(1)}(R)]$ but its dependence with respect to the kink angle (to the first order) and to the loading was made explicit. This is also feasible, in the most general case, using the present approach; the derivation is given in Appendix B.

8. CONCLUSION

We will finally outline how the results derived above can be used for numerical predictions of crack paths, propagation being supposed to obey the principle of local symmetry (PLS) of Goldstein and Salganik (1974). (This criterion has been used by numerous authors and will be fully justified in the subsequent papers.)

According to the PLS, $k_2(s)$ is zero along the whole propagation path. One must therefore equate to zero the successive terms k_2^*, $k_2^{(1/2)}\sqrt{s}$, $k_2^{(1)}s\ldots$ of the expansion of $k_2(s)$ in powers of s. Using eqns (9), (18) and (25) for $\mathbf{k}^*$, $\mathbf{k}^{(1/2)}$ and $\mathbf{k}^{(1)}$, this will yield the values of the parameters m, a^* and C^* characterizing the shape of the future crack extension, in terms of $\mathbf{k}$, T, $\mathbf{Z}$, $\mathbf{b}$. It is thus possible to predict the crack path by a step-by-step method, each step involving a numerical evaluation of $\mathbf{k}$, T, $\mathbf{Z}$, $\mathbf{b}$ and an extension of the crack by a remeshing procedure.

One important drawback of this method is that the numerical evaluation of T and $\mathbf{b}$ is a rather difficult task if a good accuracy is asked for, not to mention that of $\mathbf{Z}$ which is even harder, whichever expression of this quantity (its definition as the principal part of $\mathscr{L}^*[m, R, C, \mathscr{I}^{(1)}(R)]$ or the detailed expression (B5) derived in Appendix B) is employed.†

It is possible in this respect to give other expressions of $\mathbf{k}^{(1/2)}$ and $\mathbf{k}^{(1)}$ more suitable for practical purposes. Indeed, for a *straight* extension, eqn (18) for $\mathbf{k}^{(1/2)}$ reduces to

$$[\mathbf{k}^{(1/2)}]^{\pi m}_{a^*=0} = T\mathbf{G}(m).$$

Hence $\mathbf{k}^{(1/2)}$ can be written in the general case under the form

$$\mathbf{k}^{(1/2)} = [\mathbf{k}^{(1/2)}]^{\pi m}_{a^*=0} + a^*\mathbf{H}(m)\cdot\mathbf{k}. \tag{26}$$

The numerical evaluation of T can thus be replaced by that of $[\mathbf{k}^{(1/2)}]^{\pi m}_{a^*=0}$, which can easily be performed by comparing the initial SIFs with those at the tip of a short straight extension in the direction πm (πm being previously determined from the condition $k_2^* = 0$). In the same way, if we consider a crack extension having a zero C^*, eqn (25) for $\mathbf{k}^{(1)}$ reads

$$[\mathbf{k}^{(1)}]^{\pi m, a^*}_{C^*=0} = \mathbf{Z} + \mathbf{I}(m)\cdot\mathbf{b} + C\mathbf{J}(m)\cdot\mathbf{k} + a^*T\mathbf{K}(m) + a^{*2}\mathbf{L}(m)\cdot\mathbf{k}.$$

Let us now introduce a non-zero C^*: since $\mathscr{L}^*$ and $\mathscr{I}^{(1)}$ have been proved in Sections 4 and 5 to be independent of this parameter, this does not change the value of $\mathbf{Z}$ which is the principal part of $\mathscr{L}^*[m, R, C, \mathscr{I}^{(1)}(R)]$ for $R \to 0$, nor that of the other terms in the right-hand side of this equality. Thus (25) can be rewritten as

$$\mathbf{k}^{(1)} = [\mathbf{k}^{(1)}]^{\pi m, a^*}_{C^*=0} + C^*\mathbf{M}(m)\cdot\mathbf{k}. \tag{27}$$

The numerical evaluation of $\mathbf{Z}$ is then replaced by that of $[\mathbf{k}^{(1)}]^{\pi m, a^*}_{C^*=0}$ which can be done by comparing the initial SIFs with those at the tip of a small extension having the values of πm and a^* determined previously from the conditions $k_2^* = 0$ and $k_2^{(1/2)} = 0$, but a zero C^*.

It should be noted that in this approach, the non-universal character of the expression of $\mathbf{k}^{(1)}$ is not an important disadvantage: indeed the non-universal quantity $[\mathbf{k}^{(1)}]^{\pi m, a^*}_{C^*=0}$ can always be computed numerically with relative ease in each particular case, the essential point being that it is independent of the parameter C^* which is unknown *a priori*.

The use of this method requires, of course, knowledge of the detailed form of the universal functions $\mathbf{F}(m)$, $\mathbf{H}(m)$, $\mathbf{M}(m)$ involved in the determination of πm, a^* and C^* from eqns (9), (26) and (27) and conditions $k_2^* = 0$, $k_2^{(1/2)} = 0$, $k_2^{(1)} = 0$. The incomplete knowledge of the functions $\mathbf{H}(m)$ and $\mathbf{M}(m)$ was precisely one of the main drawbacks in Sumi's (1986a, b) studies of crack paths in configurations of practical interest.‡ The complete determination of these functions will be the subject of the next paper.

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† The use of eqn (B5) does not make the task much easier because of the complexity of the definition of the $\bar{k}_{pq}$s (see Appendix B; Sumi *et al.* (1983) have given another interpretation of the $\bar{k}_{pq}$s much better fitted for numerical applications, but unfortunately limited to the case of a *straight* initial *edge* crack).

‡ The other one being that, as mentioned in the Introduction, the formulas this author used for $\mathbf{k}^*$, $\mathbf{k}^{(1/2)}$ and $\mathbf{k}^{(1)}$ are not of fully general applicability.

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APPENDIX A

The aim of this Appendix is (i) to show the necessity of considering crack extensions with shape characterized by eqn (1) and (ii) to justify the form (eqns (14) and (19)) of the expansion of $\mathscr{G}$ in powers of s. (These questions are strongly tied together, as will be seen.) For this we will suppose that the expansion of y_2 in powers of y_1 and that of $\mathscr{G}$ in powers of s are of the form

$$y_2 = \gamma^* y_1^{1+\alpha} + o(y_1^{1+\alpha});\dagger \tag{A1}$$

$$\mathscr{G}(m, R, C, \gamma^*, s, \cdot) = \mathscr{G}^*(m, R, C, \cdot) + \mathscr{G}^{(\beta)}(m, R, C, \gamma^*, \cdot)s^\beta + o(s^\beta), \tag{A2}$$

with $0 < \alpha \leqslant 1/2$ and $0 < \beta \leqslant 1/2$ (and of course $\gamma^* \neq 0$, $\mathscr{G}^{(\beta)} \neq 0$), and show that necessarily $\alpha = \beta = 1/2$. A similar reasoning can be made to show that if these expansions contain terms proportional to $y_1^{1+\alpha'}$ and $s^{\beta'}$ with $1/2 < \alpha' \leqslant 1$ and $1/2 < \beta' \leqslant 1$, then $\alpha' = \beta' = 1$.

The reasoning follows the same lines as that in Section 6, replacing $\mathscr{G}^{(1/2)}$, $\sqrt{s}$, and a^* by $\mathscr{G}^{(\beta)}$, s^β and γ^*. When dimensions are multiplied by λ, γ^* is divided by λ^α. Therefore the homogeneity property of $\mathscr{G}$ reads (instead of (3)):

$$\mathscr{G}(m, \lambda R, C/\lambda, \gamma^*/\lambda^\alpha, \mathscr{I}) = \sqrt{\lambda}\mathscr{G}(m, R, C, \gamma^*, \mathscr{I}).$$

Expanding this equation in powers of s using eqn (A2), and identifying terms of order s^β, we get

$$\mathscr{G}^{(\beta)}(m, \lambda R, C/\lambda, \gamma^*/\lambda^\alpha, \mathscr{I}) = \lambda^{1/2-\beta}\mathscr{G}^{(\beta)}(m, R, C, \gamma^*, \mathscr{I}). \tag{A3}$$

The expansion of $\mathbf{k}(s)$ is readily obtained by inserting eqns (16) and (A2) into eqn (5):

$$\mathbf{k}(s) = \mathbf{k}^* + \mathbf{k}^{(\beta)}s^\beta + o(s^\beta)$$

where

$$\mathbf{k}^{(\beta)} = \mathscr{G}^{(\beta)}[m, R, C, \gamma^*, \mathscr{I}(R)].$$

Using eqn (A3) with $\lambda = 1/R$, we transform this equation into

$$\mathbf{k}^{(\beta)} = \mathscr{G}^{(\beta)}[m, 1, RC, R^\alpha\gamma^*, R^{1/2-\beta}\mathscr{I}(R)]. \tag{A4}$$

Let us now assume that $\beta < \alpha$. Expanding $\mathscr{G}^{(\beta)}(m, 1, RC, R^\alpha\gamma^*, \cdot)$ in powers of R and inserting the result and eqn (8) into eqn (A4), we get

† Since Oy_1 is tangent to the crack extension at the point O (see Section 2), the expansion of y_2 in powers of y_1 cannot contain any term proportional to y_1^α with $0 \leqslant \alpha \leqslant 1$.

$$\mathbf{k}^{(\beta)} = \mathscr{L}^{(\beta)}[m, 1, 0, 0, k_p\{\mathbf{f}_p(\theta)\}]\frac{1}{R^\beta} + O(R^{\alpha-\beta}, R^{1/2-\beta}).$$

Since the left-hand side is independent of R, the divergent $R^{-\beta}$ term in the right-hand side is necessarily zero. Therefore this equality yields, in the limit $R \to 0$: $\mathbf{k}^{(\beta)} = 0$, which implies that $\mathscr{L}^{(\beta)} = 0$, in contradiction with our basic hypotheses.

Our assumption that $\beta < \alpha$ is therefore absurd, which means that the expansion of $\mathscr{L}$ in powers of s does not contain any non-constant (with respect to s) term with exponent smaller than α. Thus the first non-constant term has exponent $\beta = \alpha$.

Let us then assume that $\alpha < 1/2$. Expanding eqn (A4) as above, we get

$$\mathbf{k}^{(\alpha)} = \gamma^*\frac{\partial \mathscr{L}^{(\alpha)}}{\partial \gamma^*}[m, 1, 0, 0, k_p\{\mathbf{f}_p(\theta)\}] + O(R^\alpha, R^{1/2-\alpha})$$

where the divergent $R^{-\alpha}$ term has been discarded because it is zero by the same argument as above. Taking the limit $R \to 0$, we obtain

$$\mathbf{k}^{(\alpha)} = \gamma^*\mathbf{\Phi}(m)\cdot\mathbf{k} \tag{A5}$$

where $\mathbf{\Phi}$ is a linear operator depending only on m.

We now take into account the fact that the crack extensions to be considered must be obtainable by actual propagation of the crack. Adopting the PLS as a criterion (this will be fully justified in the subsequent papers), we must equate to zero the successive terms k_2^*, $k_2^{(\alpha)}s^\alpha$ ... of the expansion of $k_2(s)$ in powers of s. The first condition, $k_2^* = 0$, yields the value of the kink angle πm. The second condition, $k_2^{(\alpha)} = 0$, yields upon use of eqn (A5):

$$\gamma^*[\Phi_{21}(m)k_1 + \Phi_{22}(m)k_2] = 0 \tag{A6}$$

where the Φ_{pq}s denote the components of $\mathbf{\Phi}(m)$. Except in exceptional cases, $\Phi_{21}(m)k_1 + \Phi_{22}(m)k_2$ has no reason to be zero for the value of m determined from the condition $k_2^* = 0$. Hence eqn (A6) implies that γ^* must be zero, in contradiction with our basic hypotheses.†

We conclude that our assumption $\alpha < 1/2$ is wrong, i.e. that $\alpha = \beta = 1/2$. Q.E.D.

APPENDIX B

The object of this Appendix is to study the term Z of eqn (25) in more detail. This will allow us (i) to substantiate the statement made in Section 7 that this term is *non-universal*; (ii) to re-derive the results of Sumi *et al.* (1983) by a different method and to extend them to fully general situations. These authors obtained, in the particular case of a straight initial edge crack with small parameters m, a^*, C^*, the following expressions of the components of Z:

$$Z_1 = [\bar{k}_{11} - (\bar{k}_{12} + \tfrac{1}{2}\bar{k}_{21})\pi m]k_1 + [\bar{k}_{12} - (\bar{k}_{11} + \tfrac{1}{2}\bar{k}_{22})\pi m]k_2 + O(m^2);$$

$$Z_2 = \left[\bar{k}_{21} + \left(\frac{\bar{k}_{11}}{2} - \bar{k}_{22}\right)\pi m\right]k_1 + \left[\bar{k}_{22} + \left(\frac{\bar{k}_{12}}{2} - \bar{k}_{21}\right)\pi m\right]k_2 + O(m^2). \tag{B1}$$

In these equations k_1 and k_2 denote as usual the initial SIFs and the $\bar{k}_{pq}$s coefficients which depend on the whole geometry of the body and the initial crack and on the partition $(\partial\Omega_t, \partial\Omega_u)$ of $\partial\Omega$, but are independent of the parameters m, a^*, C^* characterizing the crack extension shape, and also of the loading.

First we will make explicit the dependence of $\partial u_i/\partial s$ $(M, s = 0)$ with respect to m and the loading in eqn (12). Equation (9) reads, in component notation: $k_p^* = F_{pq}(m)k_q$. The $\bar{k}_{pr}^*(M)$s introduced in Section 5 are given similarly by $\bar{k}_{pr}^*(M) = F_{pq}(m)\bar{k}_{qr}(M)$ where $\bar{k}_{qr}(M)$ is defined in the same way as $\bar{k}_{qr}(M, s)$ and $\bar{k}_{qr}^*(M)$ (see Section 5), except that the SIF is to be taken at the tip of the *initial* crack. Therefore eqn (12) can be rewritten as

$$\frac{\partial u_i}{\partial s}(M, s = 0) = \frac{2(1-\nu^2)}{E}F_{p2}(m)F_{pq}(m)k_2\bar{k}_{qi}(M),$$

or, introducing the vector $\bar{\mathbf{k}}_r(M) = [\bar{k}_{1r}(M), \bar{k}_{2r}(M)]$ and dropping indications of dependence of the F_{pq}s upon m for simplicity:

$$\frac{\partial \mathbf{u}}{\partial s}(M, s = 0) = \frac{2(1-\nu^2)}{E}F_{p2}F_{pr}k_2\bar{\mathbf{k}}_r(M). \tag{B2}$$

The traction

† This phenomenon does not happen with the correct expansions (see (1), (14), (19)); indeed the second condition reads then $k_2^{(1/2)} = 0$, i.e. by eqn (18): $TG_2(m) + a^*[H_{21}(m)k_1 + H_{22}(m)k_2] = 0$, which yields generally a non-zero value for a^*.

$$\frac{\partial \mathbf{t}}{\partial s}(M,s=0) = \left[\frac{\partial \sigma_{rr}}{\partial s}(M,s=0),\ \frac{\partial \sigma_{r\theta}}{\partial s}(M,s=0)\right]$$

corresponding to the displacement field $\{\partial \mathbf{u}/\partial s(M,s=0)\}$ is deduced from this displacement by differentiation with respect to the coordinates of M, application of the elasticity operator and contraction with the vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$. This can be written symbolically $\partial \mathbf{t}/\partial s(M,s=0) = \mathbf{L}_M \cdot \partial \mathbf{u}/\partial s(M,s=0)$ where $\mathbf{L}_M$ is a linear differential operator depending on M. Application of this operator to both sides of eqn (B2) yields then, noting that in the right-hand side, the only term which depends on M is $\tilde{\mathbf{k}}_r(M)$, and incorporating the $2(1-\nu^2)/E$ factor into $\mathbf{L}_M$:

$$\frac{\partial \mathbf{t}}{\partial s}(M,s=0) = F_{wz}F_{wr}k_z\mathbf{L}_M \cdot \tilde{\mathbf{k}}_r(M).$$

$\mathscr{L}^*[m,R,C,\mathscr{F}^{(1)}(R)] = \mathscr{L}^*[m,R,C,\{\partial \mathbf{t}/\partial s(M,s=0)\}]$ can thus be written, omitting the arguments m, R, C in $\mathscr{L}^*$ for simplicity and using the linearity with respect to the force field:

$$\mathscr{L}^*[\mathscr{F}^{(1)}] = \mathscr{L}^*[\{F_{wz}F_{wr}k_z\mathbf{L}_M \cdot \tilde{\mathbf{k}}_r(M)\}] = F_{wz}F_{wr}k_z\mathscr{L}^*[\{\mathbf{L}_M \cdot \tilde{\mathbf{k}}_r(M)\}]. \quad \text{(B3)}$$

It follows from the fact that the $\tilde{k}_{tr}(M)$s are defined as SIFs at the tip of the *initial* crack that the force field $\{\mathbf{L}_M \cdot \tilde{\mathbf{k}}_r(M)\}$ is independent of the kink angle πm. Therefore the components of $\mathscr{L}^*[\{\mathbf{L}_M \cdot \tilde{\mathbf{k}}_r(M)\}] \equiv \mathscr{L}^*[m,R,C,\{\mathbf{L}_M \cdot \tilde{\mathbf{k}}_r(M)\}]$ are given, using eqn (9), by

$$[\mathscr{L}^*(\{\mathbf{L}_M \cdot \tilde{\mathbf{k}}_r(M)\})]_p = F_{pq}\tilde{k}_{qr}(R)$$

where $\tilde{k}_{qr}(R)$ denotes the qth SIF at the tip of the *initial* crack arising from the application of the force field $\{\mathbf{L}_M \cdot \tilde{\mathbf{k}}_r(M)\}$ on the boundary of the disk of radius R. Insertion of this expression into eqn (B3) yields

$$[\mathscr{L}^*(\mathscr{F}^{(1)})]_p = F_{wz}F_{wr}k_zF_{pq}\tilde{k}_{qr}(R). \quad \text{(B4)}$$

Now we know by eqn (24) that $\mathscr{L}^*[\mathscr{F}^{(1)}]$ is the sum of a divergent term proportional to R^{-1} plus another term which has a finite limit for $R \to 0$ (which we have called its *principal part* $\mathbf{Z}$). Since in the right-hand side of eqn (B4), the only term which depends on R is $\tilde{k}_{qr}(R)$, this quantity must also be the sum of a divergent term proportional to R^{-1} plus another term which has a finite limit for $R \to 0$; this limit will again be called the principal part of $\tilde{k}_{qr}(R)$ and denoted $\tilde{k}_{qr}$. Taking the principal parts of both sides of (B4), we get then:

$$Z_p = F_{pq}\tilde{k}_{qr}F_{wr}F_{wz}k_z,$$

i.e. in matrix notation ($[X]^T$ denoting the transpose of $[X]$):

$$[Z] = [F(m)][\tilde{k}][F(m)]^T[F(m)][k] \quad \text{(B5)}$$

where indications of dependence upon m have been restored.

The interpretation of the $[\tilde{k}]$ matrix in eqn (B5) can be summarized as follows: $\tilde{k}_{iq}(M)$ is the qth SIF at the initial crack tip created by a unit point force in the direction $\mathbf{e}_i$ exerted on the point M, $\partial\Omega_t/\partial\Omega_u$ being simultaneously subjected to a zero traction/displacement; $\tilde{k}_{pq}(R)$ is the pth SIF at the initial crack tip which results from the application, on the boundary of the disk of radius R, of the stresses deriving from the displacement field $u_1(M) = (2(1-\nu^2)/E)\tilde{k}_{1q}(M)$, $u_2(M) = (2(1-\nu^2)/E)\tilde{k}_{2q}(M)$; and $\tilde{k}_{pq}$ is the principal part of $\tilde{k}_{pq}(R)$ for $R \to 0$, i.e. its limit once its divergent R^{-1} part has been subtracted.

From this follows that the $[\tilde{k}]$ matrix depends on the geometry of the entire body and the initial crack and on the partition $(\partial\Omega_t, \partial\Omega_u)$ of $\partial\Omega$ (it is a *non-universal* quantity), but not on the geometric parameters m, a^*, C^* of the crack extension nor on the loading. Thus the influences of the various geometric and mechanical parameters appear as nicely separated in the expression (B5) of Z: (i) that of the loading, through the SIFs at the tip of the initial crack; (ii) that of the branching angle, through the $[F]$ and $[F]^T$ matrices; and (iii) that of the geometry of the body and the initial crack (including the partition of $\partial\Omega$), through the $[\tilde{k}]$ matrix. The third influence is in fact the only one of non-universal character.

Comparison between our result and that of Sumi *et al.* (1983) requires an expansion of (B5) to the first order in m. The first-order expression of the F_{pq}s, as given for instance by Wu (1979), is

$$F_{11}(m) = 1+O(m^2);\quad F_{12}(m) = -\frac{3\pi}{2}m+O(m^3);\quad F_{21}(m) = \frac{\pi}{2}m+O(m^3);$$

$$F_{22}(m) = 1+O(m^2).$$

Inserting these formulas into (B5), one obtains first-order expressions for the components of Z which coincide with eqns (B1) of Sumi *et al.*

In the work of Sumi *et al.*, the $\tilde{k}_{pq}$s were interpreted as the SIFs at the initial crack tip which result from the application of Bueckner's (1972) fundamental force and displacement fields (proportional to $r^{-3/2}$ and $r^{-1/2}$ respectively) on $\partial\Omega_t$ and $\partial\Omega_u$. It is possible, though somewhat intricate, to establish the correspondence between this point of view and ours. However it must be stressed that the interpretation of Sumi *et al.* is valid only in the particular case studied by these authors of a *straight* initial *edge* crack, and that the (admittedly less simple) interpretation given here is the only possible one in the general case.